An algorithm to find cospectral mates

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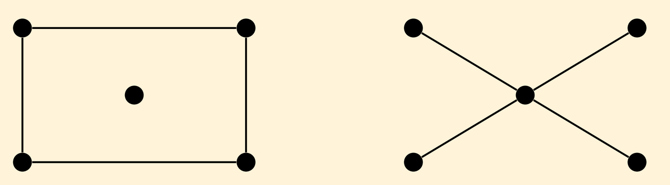
Cospectral mates

Definition. *Cospectral mates* are non-isomorphic graphs with the same spectrum (eigenvalues of the adjacency matrix).



Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$, but are not isomorphic.

Such pairs are believed to be rare:

Conjecture (Haemers, 2003).
Almost all graphs are determined by their spectrum.

How to find them?

Algebraically

Conjugating the adjacency matrix A with an orthogonal matrix Q preserves the spectrum:

$$Q^T A Q = A'$$

Combinatorially

Definition. A *switching method* is a local graph operation that produces a graph with the same spectrum. It requires:

- A subset C of the vertex set of the graph, called a *switching set*. The induced subgraph $\Gamma = G[C]$ is the *switching subgraph*.
- Adjacency conditions on the vertices ouside C.

Why care?

Switching methods are used to:

- Investigate whether a graph is determined by its spectrum, a fundamental problem in **algebraic graph theory**.
- Find **new strongly regular graphs**.
- Provide insight into **Haemers' conjecture**.
- Construct pairs of graphs that are difficult for **isomorphism testing** or **graph reconstruction**.

Known switching methods

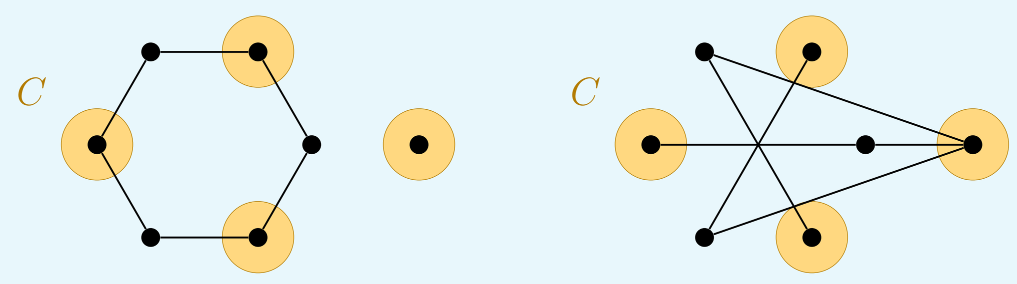
- GM-switching (Godsil and McKay, 1982)
- AH-switching (Abiad and Haemers, 2012)
- WQH-switching (Wang, Qiu and Hu, 2019)
- Sun graph switching (Mao, Wang, Liu and Qiu, 2023)
- Fano switching (Abiad, van de Berg and Simoens, 2026)
- Design switching (Ihringer and Simoens, 2026)

GM-switching

Theorem (Godsil and McKay, 1982)**.** Consider a graph G and a subset C of its vertex set such that:

- The induced subgraph on C is regular.
- Every vertex $x \notin C$ has exactly 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.

For every $x \notin C$ that has $\frac{1}{2}|C|$ neighbours in C, swap its adjacencies with C. The obtained graph is cospectral with G.

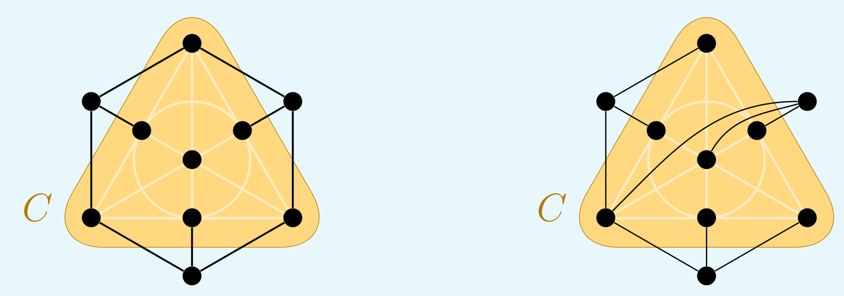


Design switching

Theorem (Ihringer and Simoens, 2026)**.** Consider a graph G and a subset C of its vertex set, identified as points of an (r, λ)-design such that

- The induced subgraph on C is edgeless or complete.
- Every vertex $x \notin C$ is adjacent to the points of a block.

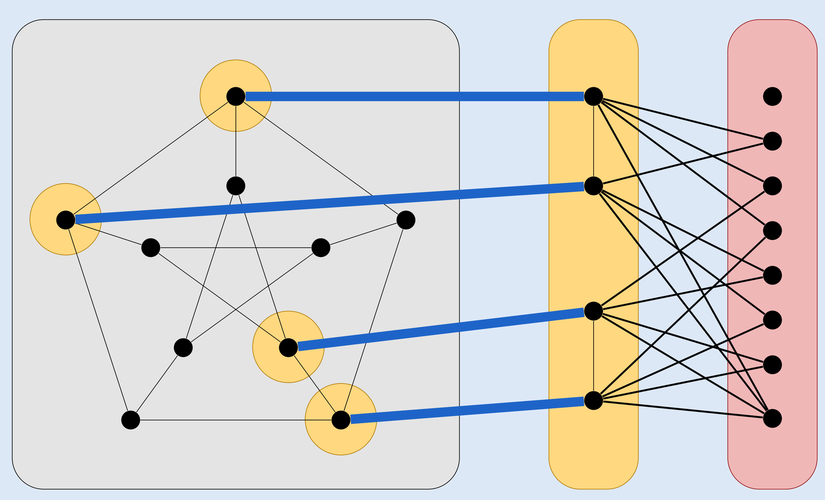
Let π be a permutation of the blocks preserving their intersection size. If $x \notin C$ is adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The obtained graph is cospectral with G.



The algorithm

Input

- Graph G
- Conditions on the switching subgraph Γ
- Adjacency conditions on the vertices outside Γ



Consecutively add edges to build a **matching** between G and Γ.

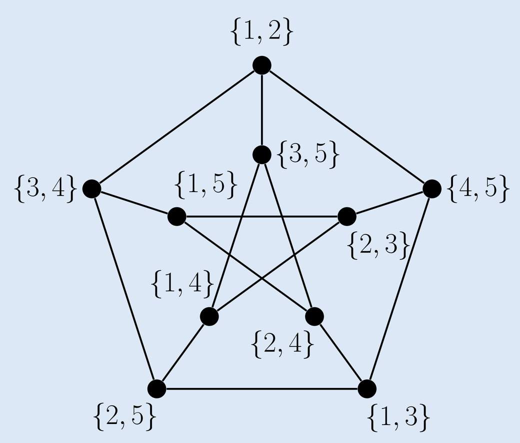
Output

- All possible switching sets C, up to symmetry.
- For each switching set, a cospectral graph. Non-isomorphism is not guaranteed, but easily checked with `nauty`.

Avoiding isomorphic situations: the **canonical construction path** method

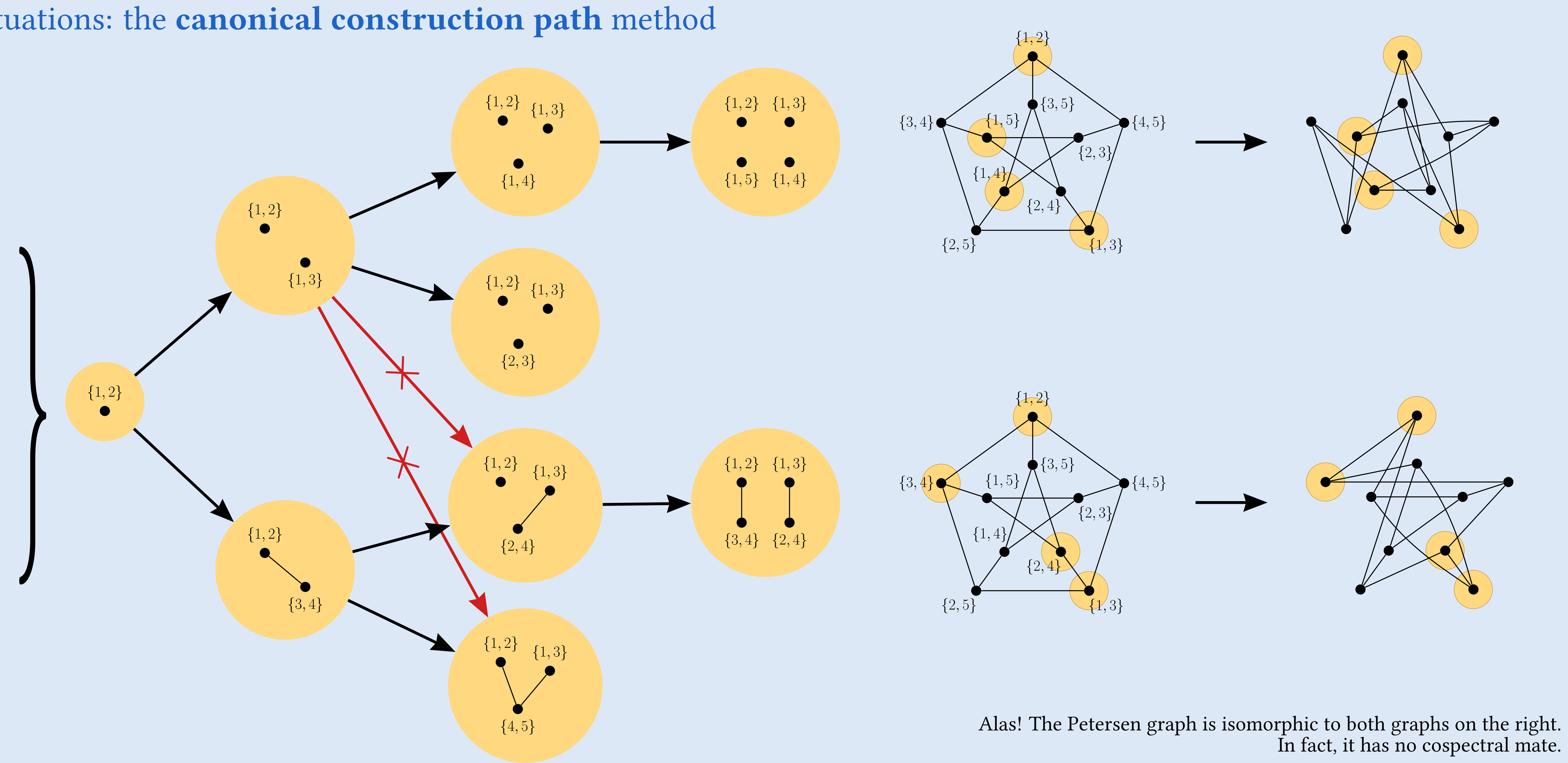
Input

- Petersen graph



GM-switching on 4 vertices:

- Regular subgraph Γ of size 4
- Vertices outside Γ have 0, 2 or 4 neighbours in Γ



Alas! The Petersen graph is isomorphic to both graphs on the right. In fact, it has no cospectral mate.

What is special?

1. The algorithm works for *any* switching method.
2. It exploits the symmetry of both the input graph G and the subgraph Γ, speeding up computations for symmetric graphs:

G	$\lvert V(G)\rvert$	$\lvert \mathrm{Aut}(G)\rvert$	method	$\lvert V(\Gamma)\rvert$	our algorithm	naive approach
CycleGraph(20)	20	40	WQH	6	4s	49s
Random G$(n = 20, p = 0.5)$	20	1	GM	6	14.6s	4.9s
CompleteGraph(30)	30	$30!$	GM	4	< 0.1s	23.5s
SymplecticPolarGraph$(6, 2)$	63	1451520	AH	10	177s	> 60m
Kneser$(9, 3)$	84	362880	WQH	6	17s	> 60m
Hamming$(4, 6)$	1296	$4!(6!)^4$	Fano	7	40s	> 60m

An application

Definition. The *q-Kneser graph* $K_q(n, k)$ has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two subspaces are adjacent if they intersect trivially.

Theorem (Simoens and Van Overberghe, 2026). All non-trivial q-Kneser graphs have a cospectral mate.

- WQH-switching
- Found with our algorithm for small values of q and generalised for all q theoretically

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